Spectral problem for dilute atomic gases using discrete Ühling-Uhlenbeck operators

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Effects of free orientations (θ which is related to the relative direction of scattering of particles with respect to the normal of the propagating plane-wave front) using four-velocity model for the dispersion relationship of ultrasound propagation in dilute atomic (hard-sphere particles of Bose and Fermi statistics) gases are presented. We address the dispersion relations thus obtained by the relevant parameter ρ (a *blocking factor*) or B which describes the Bose and Fermi particles for the quantum analog of the discrete Boltzmann system when B is positive (and $\rho=1$) and negative (and $\rho=-1$), respectively. The preliminary results show that there is no attenuation when the parameter θ equals $\pi/4$ for all Bs.

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I. INTRODUCTION

The behavior of atoms and their interactions at ultracold temperatures is a fascinating area of study. These interactions and their effects distinguish them from those encountered in collisions at room temperature [1]. Recent studies, however, have driven a tremendous growth of interest in the finite-temperature field [2,3]. Different theoretical approaches have been applied to relevant physical problems about atomic gases [4]. For instance, emerging interests in discrete and continuous models of the quantum Boltzmann equation or Ühling-Uhlenbeck equation have stimulated intensive researches recently [5–9]. Both theories and their applications are in rapid progress. There are related initial and/or boundary value problems, i.e., the former being central to the analytical or numerical approach because of the propagation of the forced sound from certain origin, while the latter being almost related to the experimental environment due to the sensors and transducers somewhere downstream, must be well defined and then solved to obtain the complex spectra or dispersion relations (real part : sound dispersion, imaginary part: sound attenuation or absorption) [7,10,11]. Our previous attempts used a fixed-orientation ($\theta=0$; θ is related to the relative direction of scattering of particles with respect to the normal of the propagating plane-wave front) four-velocity model that gave rather physical results, especially for the dilute Bose gas [7]. We noticed that Kaniadakis and Quarati have derived a nonlinear one-dimensional kinetic equation for the distribution function of particles and already extended that kinetics to D-dimensional continuous or discrete space, in order to study the distribution function of particles obeying a generalized exclusion-inclusion Pauli principle. They can obtain a general expression of the stationary distribution function, depending on the value they give to the parameter κ considering their special interests to Brownian particles (which could evolve under the action of an arbitrary external potential) [12]. (Later on, Kaniadakis also proposed a classical model for the fractional statistics which are obtained as steady states of a system of two coupled equations, linked by means of a boson-fermion transmutational potential [13].) Their approaches are more specialized than our presently used one (shall be introduced in the following section, cf. Refs. [5–9]).

In this short paper, we extend the four-velocity model to be orientation-free ($\theta\neq 0$) and then reexamine the dispersion relations (complex spectra) for the ultrasound propagation in dilute hard-sphere (monatomic) atomic gases by including dynamical correlations (by an Ühling-Uhlenbeck collision term which could describe the collision of a gas of dilute hard-sphere Fermi or Bose particles by tuning a parameter ρ [5–9] (via a *blocking factor* of the form $1+\rho f$ with f being a normalized distribution function giving the number of particles per cell, say, a unit cell, in phase space). Sound waves are presumed to be plane waves. Our preliminary results show that for $\theta=0$ and $\theta=\pi/4$, there might exist gaps of spectra for all kinds of Bose and Fermi gases considering the rarefaction measure.

II. FORMULATIONS

We assume that the gas is composed of identical particles of the same mass. The velocities of these particles are restricted to, e.g., $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$, p is a finite positive integer. The discrete number densities of particles are denoted by $N_i(\mathbf{r},t)$ [5,7,10,14,15] associated with the velocity $\mathbf{u}_i$ at point $\mathbf{r}$ and time t. If only nonlinear binary collisions are considered, considering the evolution of N_i, we have

$$\frac{\partial N_i}{\partial t}+\mathbf{u}_i\cdot\nabla N_i=F_i\equiv\sum_{j=1}^{p}\sum_{(k,l)}(A_{kl}^{ij}N_kN_l-A_{ij}^{kl}N_iN_j),$$

$$i=1,\ldots,p, \tag{1}$$

where (k,l) are admissible sets of collisions [5,7,10,14,15]. We may then define F_i of above equation for particles of Boltzmann statistics as

$$F_i(N)=\frac{1}{2}\sum_{j,k,l}(A_{kl}^{ij}N_kN_l-A_{ij}^{kl}N_iN_j), \tag{2}$$

with $i\in\Lambda=\{1,\ldots,p\}$, and the summation is taken over all $j,k,l\in\Lambda$, where A_{kl}^{ij} are nonnegative constants satisfying [5,7,10,14,15]

$$A_{kl}^{ji}=A_{kl}^{ij}=A_{lk}^{ij}$$

(indistinguishability of the particles in collision),

$$A_{kl}^{ij}(\mathbf{u}_i+\mathbf{u}_j-\mathbf{u}_k-\mathbf{u}_l)=0$$

(conservation of momentum in the collision),

$$A_{kl}^{ij}=A_{ij}^{kl} \quad \text{(microreversibility condition)}.$$

The conditions defined for the discrete velocity above require that elastic, binary collisions, such that the momentum and energy are preserved, $\mathbf{u}_i+\mathbf{u}_j=\mathbf{u}_k+\mathbf{u}_l$, $|\mathbf{u}_i|^2+|\mathbf{u}_j|^2=|\mathbf{u}_k|^2+|\mathbf{u}_l|^2$, are possible for $1\leqslant i,j,k,l\leqslant p$.

The collision operator is now simply obtained by joining A_{ij}^{kl} to the corresponding transition probability densities a_{ij}^{kl} through $A_{ij}^{kl}=S|\mathbf{u}_i-\mathbf{u}_j|\ a_{ij}^{kl}$ [5,7,10,14,15], where,

$$a_{ij}^{kl}\geqslant 0, \quad \sum_{k,l=1}^{p} a_{ij}^{kl}=1, \ \forall \ \ i,j=1,\ldots,p,$$

with S being the effective collisional cross section. If all q $(p=2q)$ outputs are assumed to be equally probable, then $a_{ij}^{kl}=1/q$ for all k and l, otherwise $a_{ij}^{kl}=0$. The term $S|\mathbf{u}_i-\mathbf{u}_j|dt$ is the volume spanned by the molecule with $\mathbf{u}_i$ in the relative motion with respect to the molecule with $\mathbf{u}_j$ in the time interval dt. Therefore, $S|\mathbf{u}_i-\mathbf{u}_j|N_j$ is the number of j molecules involved by the collision in unit time. Collisions that satisfy the conservation and reversibility conditions which have been stated above, are defined as *admissible collisions*. With the introducing of the Ühling-Uhlenbeck collision term [5–7,9] in Eqs. (1) or (2),

$$F_i=\sum_{j,k,l} A_{kl}^{ij}\ [N_kN_l(1+\rho N_i)(1+\rho N_j) -N_iN_j(1+\rho N_k)(1+\rho N_l)], \quad (3)$$

for $\rho<0$ (normally, $\rho=-1$) we obtain a gas of Fermi particles; for $\rho>0$ (normally, $\rho=1$) we obtain a gas of Bose particles, and for $\rho=0$ we obtain Eq. (1).

Considering binary collision only, from Eq. (3), the model of discrete quantum Boltzmann equation for dilute atomic (Bose and Fermi) gases proposed in Refs. [5,7] is then a system of $2n(=p)$ semilinear partial differential equations of the hyperbolic type:

$$\frac{\partial}{\partial t}N_i+\mathbf{u}_i\cdot\frac{\partial}{\partial \mathbf{x}}N_i=\frac{cS}{n}\sum_{j=1}^{2n} N_jN_{j+n}(1+\rho N_{j+1}) \times(1+\rho N_{j+n+1})^{-}2cSN_iN_{i+n} \times(1+\rho N_{i+1}) \times(1+\rho N_{i+n+1}), \quad i=1,\ldots,2n, \quad (4)$$

where $N_i=N_{i+2n}$ are unknown functions, and $\mathbf{u}_i=c(\cos[\theta+(i-1)\pi/n],\sin[\theta+(i-1)\pi/n])$; c is a reference velocity modulus, S is an effective collision cross section for the collision system, θ is the orientation starting from the positive x axis to the u_1 direction.

To study the spectral problem related to the above models, we shall study the dispersion relations for the plane wave propagating in dilute hard-sphere monatomic Bose and Fermi gases. Since passage of the sound wave causes a small departure from equilibrium resulting in energy loss owing to internal friction and heat conduction, we linearize above equations around a uniform equilibrium state (N_0) by setting $N_i(t,\mathbf{x})=N_0\ [1+P_i(t,\mathbf{x})]$, where P_i is a small perturbation. The equilibrium state here is presumed to be the same as in Refs. [7,10,11]. After some similar manipulations as mentioned in Refs. [7,10,11], with $B=\rho N_0\neq 0$ [5–7], which gives or defines the (proportional) contribution from the dilute Bose gases (if $\rho>0$, e.g., $\rho=1$) or Fermi gases ($\rho<0$, e.g., $\rho=-1$), we then have

$$\left\{\frac{\partial^2}{\partial t^2}+c^2\cos^2\left[\theta+\frac{(m-1)\pi}{n}\right]\frac{\partial^2}{\partial x^2}+4cSN_0(1+B)\frac{\partial}{\partial t}\right\}D_m =\frac{4cSN_0(1+B)}{n}\sum_{k=1}^{n}\frac{\partial}{\partial t}D_k, \quad (5)$$

where $D_m=(P_m+P_{m+n})/2$, $m=1,\ldots,n$, since $D_1=D_m$ for $1=m$ (mod$2n$). We are ready to look for the solutions in the form of plane wave $D_m=a_m\exp i(kx-\omega t)$, $(m=1,\ldots,n)$, with $\omega=\omega(k)$ [7,10,11]. This is related to the dispersion relations of one-dimensional forced ultrasound propagation of rarefied gases problem [10]. So we have

$$\left\{1+ih(1+B)-2\lambda^2\cos^2\left[\theta+\frac{(m-1)\pi}{n}\right]\right\}a_m -\frac{ih(1+B)}{n}\sum_{k=1}^{n}a_k=0, \quad m=1,\ldots,n, \quad (6)$$

with

$$\lambda=kc/(\sqrt{2}\omega), \quad h=4cSN_0/\omega\propto 1/\mathrm{Kn},$$

where h is the rarefaction parameter of the gas; Kn is the Knudsen number which is defined as the ratio of the mean free path of gases to the wave length of ultrasound.

Let $a_m=\mathcal{C}/(1+ih(1+B)-2\lambda^2\cos^2[\theta+(m-1)\pi/n])$, where $\mathcal{C}$ is an arbitrary, unknown constant, since here we only have interests in the eigenvalues of the above relation. The eigenvalue problems for different $2n$-velocity models reduce to $F_n(\lambda)=0$, or

$$1-\frac{ih(1+B)}{n}\times\sum_{m=1}^{n}\frac{1}{1+ih(1+B)-2\lambda^2\cos^2\left[\theta+\frac{(m-1)\pi}{n}\right]}=0. \quad (7)$$

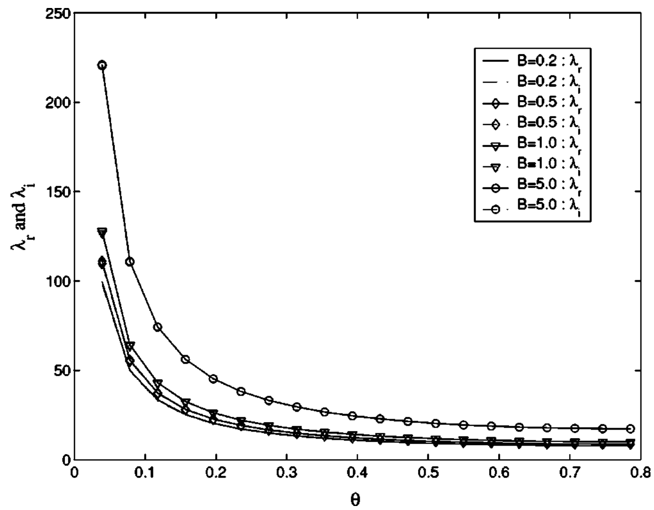

FIG. 1. Variations of λ_r and λ_i with respect to θ (rad) for $h=100$. Dilute Bose gases. λ_r denotes the dispersion and λ_i denotes the attenuation or absorption. λ_r and λ_i are dimensionless quantities. $h=4cSN_0/\omega$, S is the effective collision cross section.

We solve $n=2$, i.e., four velocity case here. The corresponding eigenvalue equations become algebraic polynomial form with the complex roots being the results of λs.

For 2×2-velocity model, we obtain

$$1-\{[ih(1+B)]/2\}\sum_{m=1}^{2} 1/\{1+ih(1+B) -2\lambda^2\cos^2 [\theta+(m-1)\pi/2]\}=0. \tag{8}$$

III. DISCUSSIONS

As $\theta\neq 0$, the degree of the complex-coefficient polynomial (equation) obtained above is now 4 instead of 2 for the fixed-orientation case ($\theta=0$) [7,10,11]. The complex-root-finding procedure thus becomes much more complicated than before. For one extreme case, i.e., the rarefaction parameter is equal to zero ($h=0$, near the vacuum limit which is surely within collisionless regime), we have roots

$$\lambda^2=\frac{1}{2\cos^2\theta}=\frac{1}{2\sin^2\theta}. \tag{9}$$

The only possible case lies at $\theta=\pm\pi/4$ and there is no attenuation ($\lambda_i=0$). For the other extreme case, $h\to\infty$, if $\theta=0$, we recover those results of the hydrodynamical or continuum-mechanic limit for dilute Bose and Boltzmann gases [7,10,11]. If $\theta\neq 0$, however, as the analysis becomes much more complicated or difficult, we should resolve it by using numerical approach (say, for the case of $h=100$) and plot the results (one branch for both real and imaginary parts of the roots) into Fig. 1. We shall focus on the variation of the spectra (λ_r and λ_i) with respect to the free orientation θ and the blocking parameter *B*. There is an essential singularity near $\theta=0$. The other (smaller) branch has trivial results ($\lambda_r\sim 1.0$ and $\lambda_i\sim 0.0$). We only present those of θs up to $\pi/4$ as spectra of orientation effects are symmetric with respect to $\theta=\pi/4$ after our checking [10,11].

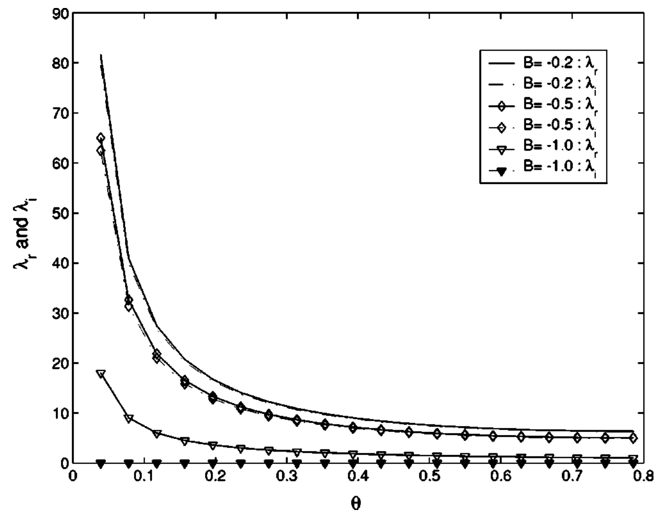

FIG. 2. Variations of λ_r and λ_i (both are dimensionless) with respect to θ (rad) for $h=100$. Cases of dilute Fermi gases ($B<0$).

We can observe that, as the (proportional) contribution from the dilute Bose gases: *B* increases, the larger branch (propagation of diffusion mode) [10,11] or larger values of both λ_r and λ_i (Fig. 1) shows an increasing trend for all θs. Once θ increases, this spectra (both λ_r and λ_i) will approach to the asymptotic case $\theta=0.7853$ (near $\pi/4$) which accounts for the propagation of the diffusion mode or entropy wave as verified in Refs. [10,11].

Similar observations hold for dilute Fermi gases ($B<0$) as shown in Fig. 2. The special case is for the attenuation λ_i when *B* is equal to $^-1$ for all θs. There is also no attenuation. This observation might be relevant to the Pauli blocking effect for dilute Fermi gases [11].

To compare present orientation-free results with previous verified fixed-orientation data [7], we noticed that, even if we only show calculations for four-velocity model, the diverse scattering of angular spectra due to $\theta\neq 0$ already covers those of higher $2\times n$-velocity models for $\theta=0$. This might be the intrinsic characteristic of the coplanar discrete velocity models [5,7,10,11,14,15].

To conclude in brief, our derivations here, as they are orientation dependent, may give more clues to the understanding of the sound propagation in microscopically atomic gases [1–4], random, disordered, or granular media [16]. The anomalous behavior of the spectra (for the larger values) near $\theta=0$, which is similar to that for the possible localization of dilute hard-sphere gases of Boltzmann-statistic particles [17] (or the forward scattering of sound by the vorticity using a first Born approximation), might also be due to the *ad hoc* assumption of having an infinitely plane wave interacting with an infinitely long-range velocity field. We shall investigate in details this and the relevant issues [18]: how to simulate the exclusion principle in microscopical many-body equations of motion in our future works.

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